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ON APPROXIMATING THE SPECTRUM OF CONVOLUTION-TYPE OPERATORS, 1. WIENER–HOPF MATRICIAL INTEGRAL OPERATORS

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ABSTRACT
Let k be an inverse Fourier transform of a real valued bounded and summable function K, and let $\{\lambda_j^{(\tau)}\}_{j=1}^{\infty}$ $(\tau > 0)$ denote the eigenvalues of the Hermitian integral operator $(W_k^{(\tau)}\phi)(t) = \int_0^\tau k(t-s)\phi(s)ds$ $(\phi \in L_2(0,\tau))$. The well known Kac, Murdock and Szegö formula asserts that

$$\lim_{\tau\to\infty} \tau^{-1} \sum_{j=1}^{\infty} [\lambda_j^{(\tau)}]^s = (2\pi)^{-1} \int_{-\infty}^{\infty} [K(x)]^s dx \quad (s = 2, 3, \cdots,).$$

The main aim of the present paper is to extend this formula to the case of a complex-valued matrix function K. We achieve this extension by developing an operator approach which is valid for a wide class of convolution type operators.

Introduction

1. Let $\mathfrak{B}$ be a Banach space, A a bounded linear operator defined on $\mathfrak{B}$, T a directed set of indices, and $\{A_\tau\}$ $(\tau \in T)$ a net of bounded linear operators strongly converging to A.

The purpose of the present work is to analyse the asymptotic behavior of the spectra of A_τ in certain particular cases.

If A_τ $(\tau \in T)$ are completely continuous and converge uniformly to A, then by the well-known results of I. C. Gohberg and M. G. Krein on the stability of root multiplicities ([8], ch. 1, §4) the eigenvalues of A_τ tend "individually", in some sense, to those of A.

In the case of strong convergence, however, the situation is more complicated, and the dependence between the spectra of A_τ and A is in general much

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weaker. This is readily seen from the following example. Let A be an integral operator defined on $\mathfrak{B} = L_2(-\infty, \infty)$ by

$$(A\phi)(t) = \int_{-\infty}^{t} e^{s-t}\phi(s)ds \qquad (-\infty < t < \infty) \quad (\phi(t) \in L_2(-\infty, \infty))$$

and let

$$(A_\tau \phi)(t) = \begin{cases} \int_{-\tau}^{t} e^{s-t}\phi(s)ds & \text{when } t \in [-\tau, \tau] \\ 0 & \text{when } t \notin [-\tau, \tau] \end{cases} \qquad (\tau > 0).$$

Obviously, the operators A_τ converge strongly to A. As is well known (cf. e.g. [6], ch. I, 23), the spectrum of A coincides with the circle $|z - \frac{1}{2}| = \frac{1}{2}$ while that of the (Volterra) operator A_τ consists of the single point $z = 0$ (for all $\tau \in (0, \infty)$).

An analogous example can be constructed in the Hilbert space $\tilde{l}_2$ of two-sided sequences $\{\xi_j\}_{j=-\infty}^{\infty}$. Indeed, let V be the left shift operator in $\tilde{l}_2$: $V\{\xi_j\}_{j=-\infty}^{\infty} = \{\xi_{j+1}\}^{\infty}$ and let $V_n = P_n V P_n$, where P_n is the orthogonal projection on the linear span of the vectors $e_j = \{\delta_{jk}\}_{k=-\infty}^{\infty}$ $(j = 0, \pm 1, \cdots, \pm n)$.

Obviously, the operators V_n converge strongly to V. As is well known the spectrum of V coincides with the unit circle, while that of the operator V_n consists of the single point $z = 0$ (for all $n = 1, 2, \cdots$).

It turns out that the "average" of the spectra of A_τ may converge to the "average" of the spectrum of A in certain cases of strong convergence of A_τ to A, even when there is no "individual" convergence of the spectra. For instance, let A be an operator, generated in $\tilde{l}_2$ by a Toeplitz matrix $(a_{j-k})_{j,k=-\infty}^{\infty}$, where a_j are the Fourier coefficients of a continuous function $a(z)$ $(|z| = 1)$. As is well known the spectrum of A coincides with the curve $a(z)$ $(|z| = 1)$. Denote by $\{\lambda_j^{(n)}\}_{j=-n}^{n}$ the eigenvalues of the restriction of $A_n = P_n A P_n$ to the subspace $P_n \tilde{l}_2$. Then the above mentioned convergence of the "average" means in particular that for every $p = 1, 2, \cdots$ there holds (see e.g. [17]):

$$\lim_{n\to\infty} \frac{\sum_{j=-n}^{n} [\lambda_j^{(n)}]^p}{2n+1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} [a(e^{i\phi})]^p \, d\phi.$$

2. Let $k = k(t) \in L_1(-\infty, \infty) \cap L_2(-\infty, \infty)$ and let $K = K(x)$ be its Fourier transform. Let us denote by W_k the Wiener–Hopf integral operator generated by k in the space $L_2(0, \infty)$:

$$(W_k \phi)(t) = \int_0^{\infty} k(t-s)\phi(s)ds \qquad (t > 0),$$

and by $W_k^{(\tau)}$ the finite truncation of W_k:

$$(W_k^{(\tau)}\phi)(t) = \begin{cases} \int_0^\tau k(t-s)\phi(s)ds & \text{for } t \in [0,\tau], \\ 0 & \text{for } t \notin [0,t]. \end{cases}$$

The first profound results on the asymptotic behavior of the spectra of $W_k^{(\tau)}$ were obtained in the well-known paper of M. Kac, W. L. Murdock and G. Szegö [14] (see also [10, sec. 8.6] and [13]). They proved that if the operator W_k is self-adjoint (i.e. K is real-valued) and $m \leqq K(x) \leqq M$, then for any function $\Phi(\lambda)$ such that $\lambda^{-2}\Phi(\lambda)$ is continuous on the interval $[m;M]$ one has

$$\lim_{\tau\to\infty} \frac{\sum_{j=1}^{\infty} \Phi(\lambda_j^{(\tau)})}{\tau} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(K(x))dx, \tag{0.1}$$

where $\{\lambda_j^{(\tau)}\}_{j=1}^{\infty}$ are eigenvalues of $W_k^{(\tau)}$ $(0 < \tau < \infty)$.[†]

It turns out that formulas like (0.1) hold for some important classes of operators.[††] Certain relevant results in the case of self-adjoint operators are mentioned in U. Grenander and G. Szegö's well-known monograph [10]. P. Schmidt and F. Spitzer [21] and I. I. Hirschman, Jr. [11] obtained a generalization of these results to a particular case of non-self-adjoint operators, namely Toeplitz matrices generated by arbitrary trigonometric polynomials. Using an operator approach, in [16–18] we establish formulas like (0.1) in the case of discrete Wiener–Hopf operators and singular integral operators generated by arbitrary bounded measurable matrix functions.

The present paper deals with the integral Wiener–Hopf operators.

The first section contains auxiliary notions and results. In the second section we prove the main lemma used throughout the paper. In sections 3 and 4 we obtain theorems on the asymptotic behaviour of the spectra of finite truncations of matricial integral Wiener–Hopf operators. In particular the formula (0.1) is generalized to any complex valued matrix-function $k(t)$ with elements in $L_1(-\infty,\infty) \cap L_2(-\infty,\infty)$.

[†] Actually this result has been obtained in [14] under somewhat different conditions on K (see Remark 3.1 of this paper).

[††] In the present work we don't touch some other aspects in studying the asymptotic behavior of "finite" truncations of convolution-type operators and we refer to I. I. Hirschman's survey [12] for an extensive bibliography. Among the later works we mention H. Dym [3].

§ 1. Preliminaries

1. In the sequel we shall often use some well-known results in the theory of symmetrically normed ideals.[†] For convenience we present them in this section.

Let $L(\mathfrak{H})$ be the ring of all bounded linear operators on the separable Hilbert space $\mathfrak{H}$. If A $(\in L(\mathfrak{H}))$ is a completely continuous operator, we denote by $\lambda_j(A)$ $(|\lambda_1(A)| \geqq |\lambda_2(A)| \geqq \cdots)$ the eigenvalues of A taking account of their algebraic multiplicity.

Let A be a completely continuous operator. The eigenvalues $s_j(A)$ $(i = 1, 2, \cdots)$ of the operator $H = (A^*A)^{1/2}$ are called the s-numbers of the operator A. We note that $s_1(A) = |A|$ is the usual operator norm of A.

If the space $\mathfrak{H}$ is infinite-dimensional, we denote by γ_p $(p \geqq 1)$ the class of all completely continuous operators, for which

$$|A|_p = \left(\sum_{j=1}^{\infty} s_j^p(A)\right)^{1/p} < \infty.$$

The set γ_p $(p \geqq 1)$ is a two-sided ideal in the ring $L(\mathfrak{H})$, and the functional $|A|_p$ satisfies all conditions of a ring norm. In addition, the following statements hold (cf. [8], ch. 2–3):

1.1°. *If $A \in \gamma_p$, $B, C \in L(\mathfrak{H})$, then*

$$|BAC|_p \leqq |B|\,|A|_p\,|C|.$$

1.2°. *If $A \in \gamma_p$ $(p \geqq 1)$, $B \in \gamma_q$ $(q = (1 - p^{-1})^{-1})$, then*

$$AB, BA \in \gamma_1 \quad \textit{and} \quad |AB|_1 \leqq |A|_p\,|B|_q.$$

1.3°. *If $A \in \gamma_1$, then for any orthonormal basis $\{\phi_j\}_{j=1}^{\infty}$ in $\mathfrak{H}$*

$$\sum_{j=1}^{\infty} (A\phi_j, \phi_j) = \sum_{j=1}^{\infty} \lambda_j(A) \overset{\text{def}}{=} \operatorname{tr} A < \infty$$

and $|\operatorname{tr} A| \leqq |A|_1$.

1.4°. *If $A \in \gamma_p$ $(p \geqq 1)$, then*

$$\sum_{j=1}^{\infty} |\lambda_j(A)|^p \leqq (|A|_p)^p.$$

We also need the following statement:

[†] See [8, Ch. 3].

1.5°. *If* $A \in \gamma_p$ $(p \geqq 1)$, *then for any* $\tau > p$

$$|A|_\tau^\tau \leqq |A|^{\tau-p} |A|_p^p.$$

Indeed

$$|A|_\tau^\tau = \sum_{j=1}^{\infty} s_j^\tau(A) \leqq s_1^{\tau-p}(A) \sum_{j=1}^{\infty} s_j^p(A) = |A|^{\tau-p} |A|_p^p.$$

The operators of the class γ_1 (called nuclear) and those of the class γ_2 (called Hilbert–Schmidt operators) are closely linked with integral operators. Let us introduce the appropriate definitions.

If $\mathfrak{B}$ is some linear set, the set of all $m \times n$ matrices with elements in $\mathfrak{B}$ is denoted everywhere in the sequel by $(\mathfrak{B})_{m\times n}$. (We denote also $(\mathfrak{B})_n = (\mathfrak{B})_{n\times 1}$.) Now let $\mathfrak{B}$ be a Banach space. Then $(\mathfrak{B})_{m\times n}$ is also a Banach space, if the norm of $A \in (\mathfrak{B})_{m\times n}$ is, for example, the sum of the norms of the components.

Similarly, if $\mathfrak{B}$ is a Hilbert space, $(\mathfrak{B})_{m\times n}$ is also a Hilbert space (if the inner product is appropriately defined).

Let $L_p(a,b)$ $(p \geqq 1)$ denote the Banach space of all complex-valued measurable functions with the norm

$$|f|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}.$$

In the Hilbert space $(L_2(a,b))_n$ we consider the matricial integral operator A given by

$$(Af)(t) = \int_a^b a(t,s) f(s) ds, \tag{1.1}$$

with the matrix function $a(t,s)$ satisfying†

$$|A|_2 = \left(\int_a^b \int_a^b |a(t,s)|_2^2 \, dt\, ds \right)^{1/2} < \infty. \tag{1.2}$$

As is well known (cf. [8], ch. 3, § 9) these are Hilbert–Schmidt operators, and conversely, any Hilbert–Schmidt operator defined in $(L_2(a,b))_n$ can be represented in the form (1.1) with some kernel $a(t,s)$ satisfying (1.2).

We shall need the following formulas for calculating the trace of integral operators (cf. [8], ch. 3, § 10).

1.6°. *If the matricial integral operator* A *is nuclear, then*

† $|a|_2^2 = \sum_{j=1}^m \sum_{k=1}^n |a_{jk}|^2$ for an $m \times n$ matrix $a = (a_{jk})$.

$$\operatorname{tr} A = \lim_{h\to 0} \frac{1}{4h^2}\int_a^b \int_a^b (2h - |t-s|)_+ \operatorname{tr} a(t,s)dtds,$$

where $y_+ = \max(y, 0)$.

If, in addition, the kernel $a(t,s)$ *is continuous, then*

$$\operatorname{tr} A = \int_a^b \operatorname{tr} a(s,s)\, ds.$$

2. In the sequel we shall need some well-known properties of the Fourier transform in the case of matrix-functions.

For convenience we shall denote henceforth $L_p(-\infty,\infty)$ by L_p.

Let $\phi = \phi(t)$ be a function in L_1 or L_2 and let $F\phi$ denote its Fourier transform: $(F\phi)(x) = \int_{-\infty}^{\infty} \phi(t)e^{ixt}dt$. If $\phi(t) = (\phi_{jk}(t))$ is an $m\times n$ matrix function in $(L_1)_{m\times n}$, then its Fourier transform $F\phi$ is the $m \times n$ matrix-function $F\phi = (F\phi_{jk})$.

Obviously all properties of a Fourier transform for scalar functions (see e.g. [22]) hold also in the case of matrix functions. In particular, for $\phi \in (L_2)_{m\times n}$ the Plancherel's classical formula becomes

$$(1.3)\qquad \int_{-\infty}^{\infty} |(F\phi)(x)|_2^2 dx = 2\pi \int_{-\infty}^{\infty} |\phi(t)|_2^2 dt.$$

We remark that when $\phi \in (L_2)_{n\times 1}$ (1.3) becomes

$$(1.4)\qquad |F\phi|^2_{(L_2)_n} = 2\pi |\phi|^2_{(L_2)_n}.$$

Now let $\phi(t) \in (L_2)_{m\times n}$, $\psi(t) \in (L_2)_{n\times l}$ or let $\phi(t) \in (L_1)_{m\times n}$, $\psi(t)\in (L_1)_{n\times l}$. The convolution $\phi * \psi$ is defined as the $m\times l$ matrix function

$$(\phi * \psi)(x) = \left(\int_{-\infty}^{\infty} \sum_{k=1}^{n} \phi_{jk}(t)\psi_{kp}(x-t)dt\right)\begin{pmatrix} j = 1,2,\cdots,m \\ p = 1,2,\cdots,l\end{pmatrix}.$$

The convolution and the Fourier transform are related by the following well-known properties (see e.g. [2, ch. 9]):

1.7°. *If* $\phi \in (L_1)_{m\times n}$ *and* $\psi \in (L_1)_{n\times l}$, *then*

$$F(\phi * \psi) = F\phi \cdot F\psi\,;\, F^{-1}(\phi * \psi) = 2\pi F^{-1}\phi \cdot F^{-1}\psi,$$

where F^{-1} *denotes the inverse Fourier transform.*

1.8°. *If* $\phi \in (L_2)_{m\times n}$ *and* $\psi \in (L_2)_{n\times l}$, *then*

$$\phi * \psi = F^{-1}(F\phi \cdot F\psi);\, \phi * \psi = 2\pi F(F^{-1}\phi \cdot F^{-1}\psi).$$

In addition we state here two simple facts, the proofs of which are left to the reader.

1.9°. *If* $\phi_j \in (L_1)_{n_{j-1}\times n_j} \cap (L_2)_{n_{j-1}\times n_j}$ $(j = 1,2,\cdots,k)$ *then*

$$\phi_1 * \phi_2 * \cdots * \phi_k \in (L_1)_{n_0\times n_k} \cap (L_2)_{n_0\times n_k}.$$

1.10°. *If* $\phi_j \in (L_1)_{n_{j-1}\times n_j} \cap (L_2)_{n_{j-1}\times n_j}$ $(j = 1,2,\cdots,k)$ $(k \geqq 2)$, *then* $\phi_1 * \phi_2 * \cdots * \phi_k$ *is continuous.*

§2. The main lemma

In the sequel we denote: $L_p(-\infty,\infty) = L_p$, $L_p(0,\infty) = L_p^+$.

Given a matrix function $k = k(t) \in (L_1)_{n\times n}$, we consider the convolution operator C_k, defined on the space $(L_2)_n$ by

$$(C_k\phi)(t) = \int_{-\infty}^{\infty} k(t-s)\phi(s)ds \qquad (-\infty < t < \infty).$$

The restriction W_k of the operator C_k to the space $(L_2^+)_n$ is called the Wiener–Hopf matricial integral operator.

Let P_τ $(0 < \tau < \infty)$ be the orthogonal projections defined on $(L_2)_n$ by the formula

$$(P_\tau\phi)(t) = \begin{cases} \phi(t), & \text{for } t \in [0,\tau] \\ 0 \quad , & \text{for } t \notin [0,\tau] \end{cases}$$

and P_τ^+ be their restriction to $(L_2^+)_n : P_\tau^+ = P_\tau \mid (L_2^+)_n$.

We also suppose $Q_\tau = I - P_\tau$, where I is the identity operator in $(L_2)_n$, $Q_\tau^+ = Q_\tau \mid (L_2^+)_n$, $C_k^\tau = P_\tau C_k P_\tau$ and $W_k^\tau = P_\tau^+ W_k P_\tau^+$.

Obviously, the subspaces $P_\tau(L_2)_n$ and $P_\tau^+(L_2^+)_n$ are isomorphic to $(L_2(0,\tau))_n$ and the restriction of W_k^τ or C_k^τ to these subspaces is a matricial integral operator $W_k^{(\tau)}$ defined on $(L_2(0,\tau))_n$ by

$$(W_k^{(\tau)}\phi)(t) = \int_0^\tau k(t-s)\phi(s)ds \qquad (0 \leqq t \leqq \tau).$$

The following statement plays an important role in proving the principal results of this paper.

LEMMA 2.1. *Given the matrix function* $k(t) \in (L_2)_{n\times n}$, *then*

$$\lim_{\tau\to\infty} \frac{| P_\tau C_k Q_\tau |_2^2}{\tau} = \lim_{\tau\to\infty} \frac{| Q_\tau C_k P_\tau |_2^2}{\tau} = 0.$$

PROOF. $Q_\tau C_k P_\tau$ may be represented as an integral operator defined on $(L_2)_n$ by

$$(Q_\tau C_k P_\tau \phi)(t) = \int_{-\infty}^{\infty} k_\tau(t,s)\phi(s)ds \qquad (-\infty < t < \infty),$$

where the kernel $k_\tau(t,s)$ is the following:

$$k_\tau(t,s) = \begin{cases} k(t-s) & \text{for } (t;s) \in D_\tau \\ 0 & \text{for } (t;s) \notin D_\tau \end{cases}$$

and D_τ is

$$D_\tau = \{(t;s) \mid 0 \leqq s \leqq \tau, \quad \text{and} \quad t \leqq 0 \quad \text{or} \quad t \geqq \tau\}.$$

According to (1.2)

$$| Q_\tau C_k P_\tau |_2^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} | k_\tau(t;s) |_2^2 dt\, ds = \int_{D_\tau} \int | k(t-s) |_2^2 dt\, ds.$$

Substituting the variables, we obtain:

$$(2.1) \qquad | Q_\tau C_k P_\tau |_2^2 = \int_{-\tau}^{\tau} | u | \, | k(u) |_2^2 du + \tau \int_{\tau}^{\infty} (| k(u) |_2^2 + | k(-u) |_2^2) du.$$

It can easily be checked that

$$\frac{1}{\tau} \int_{-\tau}^{\tau} u \, | k(u) |_2^2 du \to 0$$

if $\tau \to \infty$ and thus the desired conclusion follows by dominated convergence from (2.1).

The second formula cited in Lemma 2.1 follows from the first and the fact that $| A |_2 = | A^* |_2$.

§3. Asymptotic distribution of the spectra of truncations of Wiener–Hopf integral operators

1. Let G be a closed set in the complex z-plane and $A(G)$ denote the algebra of all functions continuous on G and holomorphic on its interior. We denote by $A^{(P)}(G)$ the subalgebra of $A(G)$ consisting of all functions $\Phi(z) \in A(G)$ such that $\Phi(z)/z^P \in A(G)$.

Let now $K(t)$ be a bounded matrix function and $K \in (L_2)_{n \times n}$. We denote by $\lambda_j(K(t))$ $(j = 1, 2, \cdots, n)$ the eigenvalues of $K(t)$ for given t and consider a compact set G containing all the points $\lambda_j(K(t))$ $(j = 1, 2, \cdots, n; \ -\infty < t < \infty)$, such that its complement is connected. Obviously, $0 \in G$. For any arbitrary $\Phi(z) \in A(G)$, the corresponding function of the matrix $K(t)$ may be undetermined in the usual sense (e.g. [4], ch. 5) and therefore we define

$$\operatorname{tr}\Phi(K(t)) = \sum_{j=1}^{n} \Phi(\lambda_j(K(t)). \tag{3.1}$$

Obviously, for sufficiently smooth functions (such as polynomials), the functional $\operatorname{tr}\Phi(K(t))$ is identical with the usual trace of the matrix $\Phi(K(t))$. Moreover, if $\Phi(z) \in A^{(2)}(G)$ is a polynomial, then $\Phi(K(t)) \in (L_1)_{n\times n}$, hence also $\operatorname{tr}\Phi(K(t)) \in L_1$. It follows from this that $\operatorname{tr}\Phi(K(t)) \in L_1$ for any $\Phi(z) \in A^{(2)}(G)$. In particular if $k = k(t) \in (L_1)_{n\times n} \cap (L_2)_{n\times n}$ and $K = Fk$, then $\operatorname{tr}\Phi(K(t)) \in L_1$ for any $\Phi \in A^{(2)}(G)$.

We denote by $\{\lambda_j^{(\tau)}\}_{j=1}^{\infty}$ the sequence of all eigenvalues of the operator $P_\tau^+ W_k P_\tau^+$ (or of $P_\tau C_k P_\tau$ which coincides with the previous) for given τ. The asymptotic behavior of the net $\{\lambda_j^{(\tau)}\}_{j=1}^{\infty}$ $(\tau > 0)$ is characterized by the following theorem.

THEOREM 3.1. *Let W_k be the Wiener–Hopf operator generated by the matrix function $k \in (L_1)_{n\times n} \cap (L_2)_{n\times n}$. Let G be a compact set containing the points $\lambda_j(K(x))$ $(j = 1, 2, \cdots, n;\ -\infty < x < \infty)$ and the eigenvalues $\lambda_j^{(\tau)}$ $(j = 1, 2, \cdots;\ 0 < \tau < \infty)$ of the operators $P_\tau^+ W_k P_\tau^+$ and such that its complement is connected.*

Then, for any function $\Phi(z) \in A^{(2)}(G)$ we have

$$\lim_{\tau\to\infty} \frac{\sum_{j=1}^{\infty} \Phi(\lambda_j^{(\tau)})}{\tau} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}(K(t))dt. \tag{3.2}$$

PROOF. We prove first (3.2) for the case $\Phi(z) = z^s$ $(s = 2, 3, \cdots)$. In this case (3.2) may be rewritten as

$$\lim_{\tau\to\infty} \frac{\operatorname{tr}(P_\tau C_k P_\tau)^s}{\tau} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}(K(t)^s)\,dt \qquad (s = 2, 3, \cdots). \tag{3.3}$$

We state now that the following relations hold:

$$(P_\tau C_k P_\tau)^m - P_\tau C_k^m P_\tau \doteq -\sum_{j=1}^{m-1} (P_\tau C_k P_\tau)^{j-1} P_\tau C_k Q_\tau C_k^{m-j} P_\tau \quad (m = 2, 3, \cdots). \tag{3.4}$$

Indeed, for $m = 2$ we have

$$(P_\tau C_k P_\tau)^2 = P_\tau C_k^2 P_\tau - P_\tau C_k Q_\tau C_k P_\tau,$$

since $P_\tau = I - Q_\tau$.

If (3.4) is valid for $m - 1$, then

$$(P_\tau C_k P_\tau)^m = P_\tau C_k P_\tau (P_\tau C_k P_\tau)^{m-1}$$
$$= P_\tau C_k P_\tau C_k^{m-1} P_\tau - \sum_{j=1}^{m-2} (P_\tau C_k P_\tau)^j P_\tau C_k Q_\tau C_k^{m-1-j} P_\tau.$$

Recalling that $P_\tau = I - Q_\tau$ and replacing the summation index j by $j+1$ we obtain (3.4).

Let us now estimate each term in the right member of (3.4). We note first that $(C_k)^r = C_{k^{r*}}$, where by 1.9° the matrix function

$$k^{r*} = \underbrace{k * k * \dots * k}_{r \text{ times}} \in (L_1)_{n\times n} \cap (L_2)_{n\times n}.$$

Therefore from Lemma 2.1 it follows that

$$\lim_{\tau\to\infty} \frac{|Q_\tau C_k^r P_\tau|_2^2}{\tau} = \lim_{\tau\to\infty} \frac{|Q_\tau C_{k^{r*}} P_\tau|_2^2}{\tau} = 0 \qquad (r = 1, 2, \cdots). \tag{3.5}$$

Using the inequality

$$\tau^{-1} |(P_\tau C_k P_\tau)^{j-1} P_\tau C_k Q_\tau C_k^{m-j} P_\tau|_1 \leqq |C_k|^{j-1} \cdot \frac{|P_\tau C_k Q_\tau|_2}{\tau^{1/2}} \cdot \frac{|Q_\tau C_k^{m-j} P_\tau|_2}{\tau^{1/2}},$$

from (3.5) and the Lemma we obtain

$$\lim_{\tau\to\infty} \tau^{-1} \left| \sum_{j=1}^{m-1} (P C_k P)^{j-1} P_\tau C_k Q_\tau C_k^{m-j} P_\tau \right|_1 = 0$$

whence by 1.3°

$$\lim_{\tau\to\infty} \tau^{-1} (\operatorname{tr} (P_\tau C_k P_\tau)^m - \operatorname{tr} P_\tau C_k^m P_\tau) = 0 \qquad (m = 2, 3, \cdots). \tag{3.6}$$

As $P_\tau C_k P_\tau$ is a Hilbert–Schmidt operator, all $(P_\tau C_k P_\tau)^m$ $(m = 2, 3, \cdots)$ are nuclear operators. Similarly the operators $P_\tau C_k Q_\tau C_k^{m-j} P_\tau$ $(j = 1, 2, \cdots, m-1;$ $m = 2, 3, \cdots)$ are nuclear, being products of Hilbert–Schmidt operators $P_\tau C_k Q_\tau$ and $Q_\tau C_k^{m-j} P_\tau = Q_\tau C_{k^{(m-j)*}} P$.

We thus conclude from (3.4) that the operators $P_\tau C_k^m P_\tau = P_\tau C_{k^{m*}} P_\tau$ are nuclear.

To sum up, the operators $P_\tau X_{k^{m*}} P_\tau \mid (L_2(0;\tau))_n$ $(m = 2, 3, \cdots)$ are nuclear operators, whose kernels $k^{m*}(x)$ are, by 1.10°, continuous. Hence, by 1.6°

$$\operatorname{tr} P_\tau C_{k^{m*}} P_\tau = \operatorname{tr}\{P_\tau C_{k^{m*}} P_\tau \mid (L_2(0;\tau))_n\} = \int_0^\tau \operatorname{tr} k^{m*}(t-t)\,dt = \tau \operatorname{tr} k^{m*}(0).$$

From 1.7° and 1.8° it follows that

$$k^{m*}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}K^m(x)dx,$$

hence

$$k^{m*}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K^m(x)dx,$$

which implies

$$\tau^{-1}\operatorname{tr} P_\tau C_{k^{m*}} P_\tau = \frac{1}{2\pi}\operatorname{tr}\int_{-\infty}^{\infty} K^m(x)\,dx.$$

The above, in conjunction with (3.6), leads to (3.3).

Thus (3.2) holds for any function $\Phi(z)$ of the form z^m $(m = 2, 3, \cdots)$, hence for any polynomial $Q(z)$ not containing terms of degree zero and one.

Now let us prove that (3.2) holds for any $\Phi(z) \in A^{(2)}(G)$.

We first note that

$$\begin{aligned}\sum_{j=1}^{\infty} |\lambda_j^{(\tau)}|^2 &\leqq |P_\tau C_k P_\tau|_2^2 = \int_0^\tau\int_0^\tau |k(s-t)|_2^2\,ds\,dt \\ &\leqq \int_{-\tau}^{\tau} (\tau - |u|)\,|k(u)|_2^2\,du \leqq \tau C,\end{aligned} \tag{3.7}$$

where $C = \int_{-\infty}^{\infty} |k(u)|_2^2\,du < \infty$, since $k(u) \in (L_2)_{n\times n}$.

On the other hand, according to (1.3) we have

$$\int_{-\infty}^{\infty}\sum_{j=1}^{n} |\lambda_j(K(x))|^2\,dx \leqq \int_{-\infty}^{\infty} |K(x)|_2^2\,dx = 2\pi C. \tag{3.8}$$

Now let $\Phi(z) \in A^{(2)}(G)$. Then by virtue of Mergelian's classical theorem [20] for given $\varepsilon > 0$ there exists a polynomial $P(z)$ such that

$$|z^{-2}\Phi(z) - P(z)| \leqq \frac{\varepsilon}{3C} \qquad (z \in G).$$

Denoting $z^2P(z) = Q(z)$, we obtain

$$|\Phi(z) - Q(z)| \leqq \frac{\varepsilon}{3C}|z|^2 \qquad (z \in G). \tag{3.9}$$

The formula (3.2) holds for $Q(z)$ since it does not contain terms of degree zero and one. Accordingly, there exists τ_0 such that

$$\left|\frac{1}{\tau}\sum_{j=1}^{\infty} Q(\lambda_j^{(\tau)}) - \frac{1}{2\pi}\int_{-\infty}^{\infty}\operatorname{tr} Q(K(x))\,dx\right| < \frac{\varepsilon}{3} \qquad (\tau > \tau_0). \tag{3.10}$$

Since $\lambda_j^{(\tau)}$, and $\lambda_j(K(x)) \in G$ it follows from (3.7)–(3.9) that

$$\frac{1}{\tau}\sum_{j=1}^{\infty} |\Phi(\lambda_j^{(\tau)}) - Q(\lambda_j^{(\tau)})| \leqq \frac{\varepsilon}{3\tau C}\sum_{j=1}^{\infty} |\lambda_j^{(\tau)}|^2 \leqq \frac{\varepsilon}{3}$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\operatorname{tr}\Phi(K(x)) - \operatorname{tr} Q(K(x))|\, dx$$

$$\leqq \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{j=1}^{n} |\Phi(\lambda_j(K(x)) - Q(\lambda_j(K(x)))|\, dx$$

$$\leqq \frac{\varepsilon}{6\pi C}\int_{-\infty}^{\infty} \sum_{j=1}^{n} |\lambda_j(K(x))|^2\, dx \leqq \frac{\varepsilon}{3}.$$

From the last two inequalities and (3.10) we obtain

$$\left| \frac{1}{\tau}\sum_{j=1}^{\infty} \Phi(\lambda_j^{(\tau)}) - \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}\Phi(K(x))dx \right| < \varepsilon$$

for all $\tau \geqq \tau_0$. Thus (3.2) holds for any $\Phi(z) \in A^{(2)}(G)$.

The theorem is proved.

REMARK 3.1. It is easy to see that Theorem 3.1 and its proof remain valid when the assumption $k \in (L_1)_{n\times n} \cap (L_2)_{n\times n}$ is replaced by the following one:

The matrix function k is the inverse Fourier transform of a bounded and absolutely integrable matrix function K ($k = F^{-1}K$).

Conditions of this type were used by M. Kac, W. L. Murdock and G. Szegö [14, 10], but it seems to us that in the context of the present paper it is preferable to impose conditions on the matrix function k than on its Fourier transform.

REMARK 3.2. From the proof of Theorem 3.1 it follows that (3.2) holds for $\Phi(z) \in A^{(1)}(G)$ provided $k(t)$ is continuous, and $P_\tau C_k P_\tau \in \gamma_1$ $(0 < \tau < \infty)$. In particular, these conditions are satisfied when $k = k_1 * k_2$, where $k_1, k_2 \in (L_1)_{n\times n} \cap (L_2)_{n\times n}$. Indeed, in this case $k(t)$ is continuous by 1.10° and since $P_\tau C_{k_j} P_\tau \in \gamma_2$ $(j = 1,2)$, $P_\tau C_{k_1} Q_\tau$, $Q_\tau C_{k_2} P_\tau \in \gamma_2$, we have

$$P_\tau C_k P_\tau = P_\tau C_{k_1} P_\tau C_{k_2} P_\tau + P_\tau C_{k_1} Q_\tau C_{k_2} P_\tau \in \gamma_1.$$

One can attain other conditions ensuring nuclearity of $P_\tau C_k P_\tau$, for example, from the results in [8] (ch. 2, § 10).

2. From the formulation of Theorem 3.1 it is clear that the class of functions $\Phi(z)$ for which (3.2) holds is substantially dependent on the compact set G which

contains the spectrum of $P_\tau C_k P_\tau$ $(0<\tau<\infty)$ and the points $\lambda_j(K(x))$ $(j=1,2,\cdots,n;\ -\infty<x<\infty)$.

For such G we may take, for example, the closed disk $\Gamma_\rho=\{z:|z|\leqq\rho\}$.with $\rho=\sup_x|K(x)|$, where $|K(x)|$ is the norm of the linear operator generated by the matrix $K(x)$ (for given x) in the n-dimensional Hilbert space $L_2^{(n)}$ of sequences $\zeta=\{\zeta_j\}_{j=1}^n$.

Indeed, it is obvious that the eigenvalues $\lambda_j(K(x))$ lie within this disk. On the other hand, if $v\in(L_2)_n$ and $V=Fv$, then by 1.8° $C_k v=F^{-1}(KV)$ and by (1.4)

$$|C_k v|_{(L_2)_n}=\frac{1}{2\pi}|KV|_{(L_2)_n}\leqq\frac{1}{2\pi}\sup_x|K(x)|\cdot|V|_{(L_2)_n}$$

$$=\sup_x|K(x)|\,|v|_{(L_2)_n}.$$

Hence

$$|P_\tau C_k P_\tau|\leqq|C_k|\leqq\sup_x|K(x)|. \tag{3.11}$$

From (3.11) it follows, indeed, that $\lambda_j^{(\tau)}$ $(j=1,2,\cdots;0<\tau<\infty)$ lie within the disk Γ_ρ. Thus (3.2) is valid in any case for all functions $\Phi(z)$ such that $\Phi(z)/z^2$ is continuous on the disk Γ_ρ and holomorphic within it.

The above class of functions can be essentially extended in a number of cases.

Let $K(x)$ be a continuous $n\times n$ matrix function. For each fixed x, let $\Omega_K(x)$ denote the numerical range of the operator induced by the matrix $K(x)$ in the space $l_2^{(n)}$, i.e.

$$\Omega_k(x)=\{\lambda\in C:\lambda=\langle K(x)\zeta,\zeta\rangle,\zeta\in l_2^{(n)},|\zeta|=1\}.$$

Let Ω_K denote the closed convex hull of all sets $\Omega_K(x)$. Obviously for each fixed x the numbers $\lambda_j(K(x))$ $(j=1,2,\cdots,n)$ are contained in $\Omega_K(x)$; therefore the set Ω_K contains all points $\lambda_j(K(x))$ $(j=1,2,\cdots,n;\ -\infty<x<\infty)$. In addition, the following statement holds:

3.1°. *The spectra of all operators* $W_k^{(\tau)}=C_k^{(\tau)}=P_\tau C_k P_\tau\,|\,(L_2(0;\tau))_n$ $(0<\tau<\infty)$ *are contained in* Ω_K.

PROOF. Let $f(t)=\{f_j(t)\}_{j=1}^n\in(L_2)_n$ and $\overline{f(t)}=\{\overline{f_j(t)}\}_{j=1}^n$. Suppose $\Phi=Ff$. Obviously $F^{-1}\bar f=(2\pi)^{-1}\bar\Phi$. By 1.8° and Parseval's theorem, we have†

$$\langle C_k f,f\rangle=\frac{1}{2\pi}\langle K\Phi,\Phi\rangle.$$

† We use the same notation ⟨ ⟩ for the inner product in l_2^n and $(L_2)_n$.

In addition, by 1.4° $\langle f, f\rangle = (2\pi)^{-1}\langle \Phi, \Phi\rangle$, hence

$$\langle (C_k - \lambda I)f, f\rangle = (2\pi)^{-1}\langle (K - \lambda E)\Phi, \Phi\rangle,$$

where E is the n-dimensional identity matrix and I is the identity operator in $(L_2)_n$. To prove the statement, it obviously suffices to establish that every point λ not included in Ω_K is a regular point of $C_k^{(\tau)}$ for all $\tau > 0$.

Let $\lambda_0 \notin \Omega_K$. Hence for some straight line l in the complex plane, the set of points of the form $\langle K(x)\zeta, \zeta\rangle - \lambda_0$ ($|\zeta| = 1$, $-\infty < x < \infty$) and the point $\lambda = 0$ lie on opposite sides of l. Therefore there exist numbers α ($|\alpha| = 1$) and δ ($\delta > 0$) such that $\operatorname{Re} \alpha(\langle K(x)\zeta, \zeta\rangle - \lambda_0) \geqq \delta$ for any n-dimensional unit vector ζ and any $x \in (-\infty, \infty)$. Thus

$$\operatorname{Re}\left[\alpha \sum_{p,q=1}^{n} (K_{p,q}(x) - \lambda_0\delta_{p,q})\zeta_p\bar{\zeta}_q\right] \geqq \delta \sum_{p=1}^{n} |\zeta_p|^2$$

for any $x \in (-\infty, \infty)$ and $\zeta = \{\zeta_p\}_{p=1}^{n}$.

Now let $f = \{f_p\}_{p=1}^{n} \in (L_2)_n$. Then

$$|\langle (C_k - \lambda_0 I)f, f\rangle|$$

$$= (2\pi)^{-1}\left|\alpha \int_{-\infty}^{\infty} \sum_{p,q=1}^{n} (K_{p,q}(x) - \lambda_0\delta_{p,q})\Phi_q(x)\bar{\Phi}_p(x)dx\right|$$

$$\geqq (2\pi)^{-1}\delta \int_{-\infty}^{\infty} \sum_{p=1}^{n} |\Phi_p(x)|^2 dx = \delta |f|^2.$$

Accordingly, for any vector function $f \in (L_2)_n$ and any $\tau > 0$ we have

$$|\langle (P_\tau C_k P_\tau - \lambda_0 P_\tau)f, f\rangle| = |\langle (C_k - \lambda_0 I)P_\tau f, P_\tau f\rangle| \geqq \delta |P_\tau f|^2.$$

Hence the image of the operator $C_k^{(\tau)} - \lambda_0 I_\tau$ is closed and its kernel consists of the single vector $x = 0$ for all $\lambda_0 \in C \setminus \Omega_K$. The set $C \setminus \Omega_K$ is connected, and $C_k^{(\tau)} - \lambda_0 I_\tau$ is invertible for sufficiently large $\lambda_0 \in C \setminus \Omega_K$, i.e. inter alia $\operatorname{Im}(C_k^{(\tau)} - \lambda_0 I_\tau) = (L_2(0;\tau))_n$. Consequently, by virtue of the well-known theorem of M. A. Krasnoselskii and M. G. Krein on the defect numbers of a linear operator (cf. e.g. [1, 7]), $\operatorname{Im}(C_k^{(\tau)} - \lambda I_\tau) = (L_2(0;\tau))_n$ for all $\lambda \in C \setminus \Omega_K$; therefore, the operator $C_k^{(\tau)} - \lambda I_\tau$ is invertible for all $\lambda \in C \setminus \Omega_K$ and the statement is proved.

From this statement and Theorem 3.1 it follows that (3.2) is valid for any function $\Phi(z)$ such that $\Phi(z)/z^2$ is continuous on Ω_K and holomorphic in its interior.

REMARK 3.3. The connectivity condition imposed in Theorem 3.1 on the complement of the compact set G is essential. This is clear from the following example.

Let

$$k(t)=\begin{cases} e^{-t}, & \text{for } t \geqq 0, \\ 0\ , & \text{for } t<0. \end{cases}$$

Obviously, the spectrum of $P_\tau C_k P_\tau$ (which is a Volterra operator) consists of the single point $\lambda = 0$ for all $\tau \in (0,\infty)$. The values taken by the function $K(x)=(FK)(x)=(1-ix)^{-1}$ lie on the circumference $G=\{\zeta : |\zeta - \frac{1}{2}| = \frac{1}{2}\}$.

Thus, this G satisfies all the conditions in Theorem 3.1 except the connectivity of the complement and that's why (3.2) is no longer valid for any function $\Phi(z) \in A^{(2)}(G)$.

Indeed, let $\Phi(z)=z^2(z-\frac{1}{2})^{-1}$. Obviously $\Phi(z)\in A^{(2)}(G)$. We have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(K(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{K^2(x)}{K(x)-\frac{1}{2}}dx = 1 .$$

while on the other hand

$$\sum_{j=1}^{\infty} \Phi(\lambda_j^{(\tau)}) = 0 \qquad (0<\tau<\infty).$$

§4. Regions with non-connected complements

As stated at the end of the preceding chapter, non-connectivity of the complement of the compact set G in Theorem 3.1 may lead to violation of (3.2). As it turns out, however, (3.2) remains valid also for a fairly large class of compact sets whose complement is non-connected.

The present section deals with the relevant results.

1. Given a compact set G in the complex plane, we denote by G_j ($j=1,2,\cdots,\omega$)† the connected components of its complement, and by $R(G)$ the closure (in the uniform norm) of the algebra of all rational functions with poles outside of G.††

Let now $K(x)$ be a bounded matrix function with $K(x)\in(L_2)_{n\times n}$ and $\lambda_j(K(x)\in G$ ($j=1,2,\cdots,n$; $-\infty<x<\infty$), where G is a compact set such that $A(G)=R(G)$.

For a point $a \notin G$, the inequality

† ω may take on finite or infinite values.

†† The necessary and sufficient conditions for $A(G)=R(G)$ were established by A. G. Vituškin [23, 24].

$$|\det(K(x)-aE)| = \prod_{j=1}^{n} |\lambda_j(K(x))-a| \geqq \rho^n(a,G) \tag{4.1}$$

is obviously true, where $\rho(a, G)$ is the distance from a to G, and this implies that $(K - aE)^{-1}$ is a bounded function of x.

For $\Phi(z) \in A(G)$ the functional $\operatorname{tr}\Phi(K(x))$ is defined by (3.1). From (4.1) it readily follows that for a rational function $\Phi(z) \in A^{(2)}(G)$ we have $\Phi(K(x)) \in (L_1)_{n\times n}$, hence also $\operatorname{tr}\Phi(K(x)) \in L_1$. From this it is easy to obtain that $\operatorname{tr}\Phi(K(x)) \in L_1$ for any $\Phi(z) \in A^{(2)}(G)$.

We denote by $C_k^{(\tau)}$ the restriction of the operator $P_\tau C_k P_\tau$ to the subspace $P_\tau(L_2)_n$ and by I_τ the identity operator in this subspace.

THEOREM 4.1. *Let $k = k(t) \in (L_1)_{n\times n} \cap (L_2)_{n\times n}$ and G be a compact set containing the points $\lambda_j(K(x))$ $(j = 1,2,\cdots,n; -\infty < x < \infty)$ and $\lambda_j^{(\tau)}$ $(j = 1,2,\cdots; 0 < \tau < \infty)$, and such that $A(G) = R(G)$.*

If in each of the connected domains G_j $(j = 1,2,\cdots,\omega)$ complementary to G, there exists at least one point a_j such that

$$\sup_\tau |(C_k^{(\tau)} - a_j I_\tau)^{-1}| < \infty \quad (j = 1,2,\cdots,\omega),$$

then (3.2) *holds for any function $\Phi(z) \in A^{(2)}(G)$.*

PROOF. We note first that it will be sufficient to prove the formula (3.2) for functions of the form $\Phi(z) = z^m$ $(m = 2,3,\cdots)$ and $\Phi(z) = z^2(z - a_l)^{-m}$ $(m = 1,2,\cdots; l = 1,2,\cdots,\omega)$ and then it can be extended to any function $\Phi(z) \in A^{(2)}(G)$ as in Theorem 3.1. For $\Phi(z) = z^m$ $(m = 2,3,\cdots)$ the formula (3.2) has already been established so that it suffices to prove the relations

$$\lim_{\tau\to\infty} \frac{\sum_{j=1}^{\infty} (\lambda_j^{(\tau)})^2(\lambda_j^{(\tau)} - a_l)^{-m}}{\tau} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr} K^2(x)(K(x) - a_l E)^{-m}\, dx$$

$$(m = 1,2,\cdots; l = 1,2,\cdots,\omega)$$

which may be rewritten in the form

$$\lim_{\tau\to\infty} \frac{\operatorname{tr}(C_k^{(\tau)})^2(C_k^{(\tau)} - a_l I_\tau)^{-m}}{\tau} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr} K^2(x)(K(x) - a_l E)^{-m}\, dx \tag{4.2}$$

$$(m = 1,2,\cdots; l = 1,2,\cdots\omega).$$

To prove (4.2) we fix some l and consider the matrix function

$$\hat{K} = (K - a_l E)^{-1} + a_l^{-1} E = a_l^{-1} K (K - a_l E)^{-1}.$$

Since $k \in (L_1)_{n\times n} \cap (L_2)_{n\times n}$, $K = Fk$ is bounded and $K \in (L_2)_{n\times n}$. Hence $\hat{K}$ is also bounded and $\hat{K} \in (L_2)_{n\times n}$. Denoting $\hat{k} = F^{-1}\hat{K}$ it is obvious that $\hat{k} \in (L_2)_{n\times n}$.

We claim that the operator $C_{\hat{k}}$ is defined and bounded in $(L_2)_n$.

Indeed let $\phi \in (L_2)_n$. We denote $\Phi = F\phi \in (L_2)_n$. By 1.8° $C_{\hat{k}}\phi = F^{-1}(\hat{K}\Phi)$. Since $\hat{K}$ is bounded, the vector function $\hat{K}\Phi \in (L_2)_n$, whence $F^{-1}(\hat{K}\Phi) \in (L_2)_n$. In addition we obtain in analogy to (3.11)

$$|C_{\hat{k}}\phi|_{(L_2)_n} \leqq \sup_x |\hat{K}(x)|\,|\phi|_{(L_2)_n}.$$

Using the operator $C_{\hat{k}}$ one can construct the inverse operator for $C_k^{(\tau)} - a_l I_\tau$.

Indeed, since

$$K^2(K - a_l E)^{-1} - a_l K(K - a_l E)^{-1} - K \equiv 0$$

we obtain by 1.8° that $k * \hat{k} - a_l \hat{k} - a_l^{-1} k = 0$. Hence

$$(C_k - a_l I)(C_{\hat{k}} - a_l^{-1} I) = C_{k*\hat{k} - a_l\hat{k} - a_l^{-1}k} + I = I. \tag{4.3}$$

Multiplying both members of (4.3) by P_τ from the left and the right we obtain

$$(P_\tau C_k P_\tau - a_l P_\tau)(P_\tau C_{\hat{k}} P_\tau - a_l^{-1} P_\tau) = P_\tau - P_\tau C_k Q_\tau C_{\hat{k}} P_\tau,$$

since $P_\tau + Q_\tau = I$.

Considering the restriction of this equality to $P_\tau(L_2)_n$, we have

$$(C_k^{(\tau)} - a_l I_\tau)(C_{\hat{k}}^{(\tau)} - a_l^{-1} I_\tau) = I_\tau - P_\tau C_k Q_\tau C_{\hat{k}} P_\tau \mid P_\tau(L_2)_n.$$

The conditions of the theorem guarantee that $C_k^{(\tau)} - a_l I_\tau$ is invertible and in view of the last formula we have

$$(C_k^{(\tau)} - a_l I_\tau)^{-1} = C_{\hat{k}}^{(\tau)} - a_l^{-1} I_\tau + M_\tau, \tag{4.4}$$

where

$$M = (C_k^{(\tau)} - a_l I_\tau)^{-1} P_\tau C_k Q_\tau C_{\hat{k}} P_\tau \mid P_\tau(L_2)_n.$$

Obviously,

$$|M_\tau|_1 \leqq |(C_k^{(\tau)} - a_l I_\tau)^{-1}|\,|P_\tau C_k Q_\tau|_2\,|Q_\tau C_{\hat{k}} P_\tau|_2.$$

The last inequality and Lemma 2.1 imply that

$$\lim_{\tau\to\infty} \tau^{-1}|M_\tau|M_1 = 0 \tag{4.5}$$

since $\sup_\tau |(C_k^{(\tau)} - a_l I_\tau)^{-1}| < \infty$.

In addition, it is obvious that

(4.6) $$\sup_{\tau} |M_\tau| < \infty.$$

We now introduce the following notation:

$$\hat{K}_m = K^2(K - a_l E)^{-m}, \quad \hat{k}_m = F^{-1}\hat{K}_m \qquad (m = 1, 2, \cdots).$$

As in the case of $\hat{k}$, one can easily show that the operators $C_{\hat{k}_m}$ are defined and bounded in $(L_2)_n$ for all $m = 1, 2, \cdots$.

We will prove by induction that

(4.7) $$(C_k^{(\tau)})^2(C_k^{(\tau)} - a_l I)^{-m} = C_{\hat{k}}^{(\tau)} + Z_m^{(\tau)} \qquad (m = 1, 2, \cdots),$$

where the operators $Z_m^{(\tau)}$ satisfy

(4.8) $$\lim_{\tau\to\infty} \tau^{-1}|Z_m^{(\tau)}|_1 = 0; \quad \sup_{\tau} |Z_m^{(\tau)}| < \infty \qquad (m = 1, 2, \cdots).$$

If $m = 1$, we multiply both sides of (4.4) by a_l^2 and add to them the operator $a_l I_\tau + C_k^{(\tau)}$. The left side then becomes

$$a_l I_\tau + C_k^{(\tau)} + a_l^2(C_k^{(\tau)} - a_l I_\tau)^{-1} = (C_k^{(\tau)})^2(C_k^{(\tau)} - a_l I_\tau)^{-1}$$

and the right side

$$a_l^2 C_{\hat{k}}^{(\tau)} + C_k^{(\tau)} + a_l^2 M_\tau = C_{a_l^2\hat{k}}^{(\tau)} + a_l^2 M_\tau.$$

Since

$$F^{-1}\hat{K}_1 = F^{-1}(K^2(K - a_l E)^{-1}) = F^{-1}(a_l E + a_l^2(K - a_l E)^{-1} + K) = a_l^2\hat{k} + k,$$

we have $\hat{k}_1 = a_l^2 \hat{k} + k$.

Equation (4.4) becomes, after the above transformations,

$$(C_k^{(\tau)})^2(C_k^{(\tau)} - a_l I_\tau)^{-1} = C_{\hat{k}_1}^{(\tau)} + Z_1^{(\tau)},$$

where $Z_1^{(\tau)} = a_l^2 M_\tau$ satisfies (4.8) in view of (4.5) and (4.6).

The statement is thus proved for $m = 1$.

Suppose now that Eqs. (4.7), (4.8) hold for some m. We will prove them for $m + 1$. For this purpose we multiply correspondingly the left and the right sides of (4.7) and (4.4).

We obtain

(4.9) $$(C_k^{(\tau)})^2(C_k^{(\tau)} - a_l I_\tau)^{-(m+1)} = C_{\hat{k}_m}^{(\tau)} C_{\hat{k}}^{(\tau)} - a_l^{-1} C_{\hat{k}_m}^{(\tau)} + Y_m^{(\tau)},$$

where

$$Y_m^{(\tau)} = C_{\hat{k}_m}^{(\tau)} M_\tau + Z_m^{(\tau)}(C_{\hat{k}}^{(\tau)} - a_l^{-1} I_\tau) + Z_m^{(\tau)} M_\tau.$$

From (4.5), (4.6) and (4.8) and the boundedness of $C_{\hat{k}}$ and $C_{\hat{k}_m}$ it follows that

$$\lim_{\tau\to\infty} \tau^{-1} | Y_m^{(\tau)} |_1 = 0; \qquad \sup_\tau | Y_m^{(\tau)} | < \infty. \tag{4.10}$$

Besides, we have

$$C_{\hat{k}_m}^{(\tau)} C_{\hat{k}_m * \hat{k}}^{(\tau)} + P_\tau C_{\hat{k}_m} Q_\tau C_{\hat{k}} P_\tau \,|\, (L_2)_n.$$

By the boundedness of $C_{\hat{k}_m}$ and $C_{\hat{k}}$ and Lemma 2.1, we have also

$$\lim_{\tau\to\infty} \tau^{-1} | P_\tau C_{\hat{k}_m} Q_\tau C_{\hat{k}} P_\tau \,|\, P_\tau (L_2)_n |_1 = 0;$$

(4.11)

$$\sup_\tau | P_\tau C_{\hat{k}_m} Q_\tau C_{\hat{k}} P_\tau \,|\, P_\tau (L_2)_n | < \infty.$$

Let us rewrite (4.9) as follows:

$$(C_{\hat{k}}^{(\tau)})^2 (C_{\hat{k}}^{(\tau)} - a_l I_\tau)^{-(m+1)} = C_{\hat{k}_m * \hat{k} - a_l^{-1} \hat{k}_m}^{(\tau)} + Z_{m+1}^{(\tau)}, \tag{4.12}$$

where

$$Z_{m+1}^{(\tau)} = Y_m^{(\tau)} + | P_\tau C_{\hat{k}_m} Q_\tau C_{\hat{k}} P_\tau \,|\, P_\tau (L_2)_n.$$

From the equalities

$$\hat{K}_{m+1} = K^2 (K - a_l E)^{-m+1} = K^2 (K - a_l E)^{-m} ((K - a_l E)^{-1} + a_l^{-1} E)$$
$$- a_l^{-1} K^2 (K - a_l E)^{-m} = \hat{K}_m \hat{K} - a_l^{-1} \hat{K}_m$$

it follows that $\hat{k}_{m+1} = F^{-1} \hat{K}_{m+1} = \hat{k}_m * \hat{k} - a_l^{-1} \hat{k}_m$ and (4.12) finally becomes

$$(C_{\hat{k}}^{(\tau)})^2 (C_{\hat{k}}^{(\tau)} - a_l I_\tau)^{-(m+1)} = C_{\hat{k}_{m+1}}^{(\tau)} + Z_{m+1}^{(\tau)},$$

where $Z_{m+1}^{(\tau)}$ satisfies (4.8) because of (4.10) and (4.11).

The equalities (4.7), (4.8) are thus proved.

Proposition 1.3° implies now that

$$\lim_{\tau\to\infty} \tau^{-1} \operatorname{tr} [(C_k^{(\tau)})^2 (C_k^{(\tau)} - a_l I_\tau)^{-m} C_{\hat{k}_m}^{(\tau)}] = \lim_{\tau\to\infty} \tau^{-1} \operatorname{tr} Z_m^{(\tau)} = 0. \tag{4.13}$$

On the other hand $(C_{\hat{k}}^{(\tau)})^2 \in \gamma_1$, hence $(C_{\hat{k}}^{(\tau)})^2 (C_{\hat{k}}^{(\tau)} - a_l I_\tau)^{-m} \in \gamma_1$. From (4.7) it follows then that $C_{\hat{k}_m}^{(\tau)} \in \gamma_1$. In addition, the kernel $\hat{k}_m(t)$ is the inverse Fourier transform of the matrix function $K^2 (K - a_l E)^{-m} \in (L_1)_{n\times n}$, hence by 1.6°

$$\operatorname{tr} C_{k_m}^{(\tau)} = \operatorname{tr} \int_0^\tau k_m(t-t)dt = \tau \operatorname{tr} \hat{k}_m(0) = \frac{\tau}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr} K^2(x)(K(x)-a_l E)^{-m} dx.$$

From the above and (4.13) we obtain (4.2), and the theorem is proved.

REMARK 4.1. We observe that Remarks 3.1 and 3.2 on Theorem 3.1 can easily be extended to the present theorem.

REMARK 4.2. The example in Remark 3.3 shows that the condition $\sup_\tau |(C_k^{(\tau)} - a_l I_\tau)^{-1}| < \infty$ in Theorem 4.1 is essential. Indeed, taking

$$G_1 = \{\zeta : |\zeta - \tfrac{1}{2}| < \tfrac{1}{2}\}, \qquad G_2 = \{\zeta : |\zeta - \tfrac{1}{2}| > \tfrac{1}{2}\}$$

we note that for all points $\zeta \in G_1$

$$\sup |(C_k^{(\tau)} - \zeta I_\tau)^{-1}| = \infty$$

(which readily follows from the theory of projection methods for Wiener–Hopf equations).[†] In these circumstances (3.2) is not satisfied for any function $\Phi(z) \in A^{(2)}(G)$.

2. In this section we study results on the location of the spectra of $C_k^{(\tau)}$. We then apply these results to some concrete examples of regions G.

First of all, the following remark holds.

REMARK 4.3. Given an arbitrary sequence $\{\tau_p\}_{p=1}^\infty$ of positive numbers tending to infinity, we denote by $G_{\{\tau_p\}}$ a compact set containing the points $\lambda_j(K(x))$ ($j = 1, 2, \cdots, n$; $-\infty < x < \infty$) and $\lambda_j^{(\tau_p)}$ ($j = 1, 2, \cdots; p = 1, 2, \cdots$). Suppose that $A(G_{\{\tau_p\}}) = R(G_{\{\tau_p\}})$ and that in each connected component of the complement of $G_{\{\tau_p\}}$ there exists at least one point a such that

$$\sup_{1 \leq p < \infty} |(C_k^{(\tau_p)} - aI_{\tau_p})^{-1}| < \infty.$$

In analogy to Theorem 4.1 the equality

$$\lim_{p\to\infty} \frac{\sum_{j=1}^{\infty} \Phi(\lambda_j^{(\tau_p)})}{\tau_p} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr} \Phi(K(x))dx$$

holds for any function $\Phi(z) \in A^{(2)}(G_{\{\tau_p\}})$.

We conclude that (3.2) holds for any function $\Phi(z) \in \bigcap_{\{\tau_p\}} A^{(2)}(G_{\{\tau_p\}})$, with the intersection taken over all sequences $\{\tau_p\}_{p=1}^\infty$ of positive numbers tending to infinity.

[†] For all relevant definitions and results, see [6].

We next introduce some definitions.

Let Γ be a unit circle in the complex plane. By $L_p(\Gamma)$ we denote the Banach space of all functions $f(\zeta)$ measurable on Γ and such that

$$|f(\zeta)|_p = \left(\frac{1}{2\pi}\int_0^{2\pi}|f(e^{i\phi})|^p d\phi\right)^{1/p} < \infty$$

and by H_p^+ (H_p^-) $(p \geqq 1)$ the subspace of all functions of $L_p(\Gamma)$, whose Fourier coefficients with negative (positive) indices are equal to zero.

If $A(\zeta)$ $(\zeta \in \Gamma)$ is a continuous non-singular matrix function on Γ, then according to well-known results on factorization (see, e.g. [6], ch. 8), the representations

$$A(\zeta) = A_-(\zeta)D(\zeta)A_+(\zeta); \qquad A(\zeta) = \tilde{A}_+(\zeta)\tilde{D}(\zeta)\tilde{A}_-(\zeta)$$

hold, $D(\zeta)$ and $\tilde{D}(\zeta)$ being diagonal matrix functions of the form

$$D(\zeta) = \| \zeta^{\kappa_j^r}\delta_{j,k} \|_{j,k=1}^n; \qquad \tilde{D}(\zeta) = \| \zeta^{\kappa_j^l}\delta_{j,k} \|_{j,k=1}^n,$$

$\kappa_1^r \geqq \kappa_2^r > \cdots \geqq \kappa_n^r$, $\kappa_1^l \geqq \kappa_2^l \geqq \cdots \geqq \kappa_n^l$ are some integers, and

$$(A_+(\zeta))^{\pm 1}, (\tilde{A}_+(\zeta))^{\pm 1} \in (H_p^+)_{n\times n}; \quad (A_-(\zeta))^{\pm 1}, (\tilde{A}_-(\zeta))^{\pm 1} \in (H_p^-)_{n\times n}$$

for any $p \in (1, \infty)$.

The integers κ_j^r (corr. κ_j^l) are called right (left) partial indices of the matrix function $A(\zeta)$. The set Λ_A of all points λ for which $\det(A(\zeta) - \lambda E) = 0$ for a certain ζ $(|\zeta| = 1)$ decomposes the complex plane into a series of connected components, for each of which the sum of the right (left) partial indices of $A(\zeta) - \lambda E$ remains constant for all λ.

Let now $k \in (L_1)_{n\times n}$ and $K = Fk$. Obviously, the matrix function

$$A(\zeta) = K\left(i\frac{1+\zeta}{1-\zeta}\right)$$

is defined and continuous on the unit circle Γ.

From the results of [6, ch. 8] one can easily deduce the following statement.

4.1°. *Suppose that in some connected component N, generated by Λ_A, there exist points Z_1 and Z_2 such that all right partial indices of the matrix function $A(\zeta) - Z_1E$ and all left ones of $A(\zeta) - Z_2E$ vanish. Let $\{\tau_p\}_{p=1}^{\infty}$ be any sequence of positive numbers tending to infinity. In that case all points of N except some set M of isolated points, are regular points of the operators $C_k^{(\tau_p)}$ $(p = 1, 2, \cdots)$ and at each of them we have*

$$r_Z = \sup_{1 \le p < \infty} |(C_k^{(\tau_p)} - ZI_{\tau_p})^{-1}| < \infty \qquad (Z \in N \setminus M).$$

PROOF. Since the sum of the right partial indices remains constant in each connected component generated by Λ_A, the condition on the right partial indices of $A(\zeta) - Z_1 E$ implies that the sum of all these indices of the matrix function $A(\zeta) - ZE$ vanishes for any $Z \in N$. This means (by theorem 8.6.1 in [6]) that the operator $W_k - ZI$ is a Φ-operator† with zero index for all $Z \in N$. By the same theorem the operator $W_k - Z_1 I$ is invertible. It follows, using I. C. Gohberg's [5] well known results on analytic operator functions, that the operator $W_k - ZI$ is invertible at all points of N, except for some set of isolated points M_1.

Suppose now that for a point $Z_2 \in N$, all left partial indices of $A(\zeta) - Z_2 E$ are zero. Then, obviously, all the right indices of the matrix function $\tilde{A}(\zeta) - Z_2 E$ are also zero, where $\tilde{A}(\zeta) = A(\zeta^{-1})$. By the same considerations, we obtain that the operator $W_{\tilde{k}} - ZI$ (where $\tilde{k}(t) = k(-t)$) is invertible at all points of N except for some set of isolated points M_2. We suppose $\tilde{M} = M_1 \cup M_2$.

From the invertibility of $W_k - ZI$ and $W_{\tilde{k}} - ZI$ in $N \setminus \tilde{M}$ we conclude, from theorem 8.6.2 in [6], and the uniform boundness principle, that for every point $Z \in N \setminus M$ there exists a number p_Z such that the operators $C_k^{(\tau_p)} - ZI_{\tau_p}$ are invertible for all $p \geqq p_Z$ and

$$\rho_Z = \sup_{p \geqq p_Z} |(C_k^{(\tau_p)} - ZI_{\tau_p})^{-1}| < \infty.$$

Let us now denote $M_3 = \{Z \mid Z \in N \setminus \tilde{M}, r_Z = \infty\}$ and prove that M_3 contains only isolated points.

Indeed, let $Z_0 \in M_3$ and $C(Z_0, \rho)$, be a disk with center at Z_0 and radius $\rho < \rho_{Z_0}$ such that $C(Z_0, \rho) \in N$. From the equality

$$C_k^{(\tau_p)} - ZI_{\tau_p} = (C_k^{(\tau_p)} - Z_0 I_{\tau_p})[I_{\tau_p} - (Z - Z_0)(C_k^{(\tau_p)} - Z_0 I_{\tau_p})^{-1}]$$

it follows that for all $Z \in C(Z_0, \rho)$ and $p \geqq p_{Z_0}$ the operators $C_k^{(\tau_p)} - ZI_{\tau_p}$ are invertible, and

$$\sup_{p \geqq p_{Z_0}} |(C^{(\tau_p)} - ZI_{\tau_p})^{-1}| < \infty \qquad (Z \in C(Z_0, \rho)).$$

Accordingly, the circle $C(Z_0, \rho)$ may only contain eigenvalues of $C_k^{(\tau_p)}$ for $p = 1, 2, \cdots, p_{Z_0}$. Since all these operators are completely continuous, we con-

†For definitions and results from the theory of Φ-operators, see e.g. [7, 9, 15].

clude that the circle contains only a finite number[†] of the above eigenvalues. Hence there exists a neighbourhood $U_{Z_0} \subset C(Z_0, \rho)$ containing none of them with the exception of Z_0 itself and $r_Z < \infty$ for all $Z \in U_{Z_0}$ $(Z \neq Z_0)$, i.e. Z_0 is an isolated point in M_3. We suppose $M = \tilde{M} \cup M_3$ and the statement is proved.

REMARK 4.4. By some similar considerations it is not difficult to show that if $\tilde{N}$ is a closed set contained in a component N with the same properties as in 4.1°, then there exists $\tilde{\tau}$ such that all points in $\tilde{N}$ are regular for $C_k^{(\tau)}$ at $\tau > \tau$, while

$$\sup |(C_k^{(\tau)} - ZI_\tau)^{-1}| < \infty \qquad (Z \in \tilde{N}).$$

The statement 4.1°, Remarks 4.3 and 4.4, Theorem 4.1 and finally the conditions for $R(G) = A(G)$ lead to the following statement.

4.2°. *Suppose that* $k \in (L_1)_{n\times n}$, $K = Fk$,

$$A(\zeta) = K\left(i\frac{1+\zeta}{1-\zeta}\right) \qquad (|\zeta| = 1).$$

Let G be some compact set obtained by the removal from the complex plane of a finite number of the connected components generated by Λ_A (including the unbounded component) in each of which there exists a pair of points Z_1, Z_2 such that all right (corr. left) partial indices of the matrix function $A(\zeta) - Z_1E$ (corr. $A(\zeta) - Z_2E$) are zero, and let $\tilde{G}$ be some neighbourhood of G. Then there exists τ_0 such that the set $\tilde{G}$ contains all the eigenvalues $\{\lambda_j^{(\tau)}\}_{j=1}^{\infty}$ of the operators $C_k^{(\tau)}$ for $\tau \geqq \tau_0$ and the relationships (3.2) *hold for any function $\phi(z)$, holomorphic at interior points of G and continuous in $\tilde{G}$.*

For $n = 1$ this statement is somewhat simpler. For a scalar function $A(\zeta)$, the set Λ_A is identical with the image of the unit circle $|\zeta| = 1$ mapped by the function $A(\zeta)$, and $\kappa^r = \kappa^l = \operatorname{ind} A(\zeta) = (2\pi)^{-1}[\arg A(\zeta)]_0^{2\pi}$, where $[\quad]_0^{2\pi}$ denotes the increment of the function on the segment $[0; 2\pi]$. For each connected component generated by the curve $A(\zeta)$, $\operatorname{ind}(A(\zeta) - \lambda)$ is constant for all λ.

REMARK 4.5. For statement 4.2° we have used one of the conditions which guarantee the equality $A(G) = R(G)$ (i.e. the complement of G consists of a finite number of connected components). Using other conditions (for such see [19, 23, 24]) one can obtain similar statements.

[†] The point $\lambda = 0$ obviously belongs to Λ_A, hence $0 \notin N$.

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